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High-energy localized eigenstates of an electronic resonator in a magnetic field

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Abstract
We present a semiclassical analysis of the high-energy eigenstates of an electron inside a closed resonator. An asymptotic method of the construction of the energy spectrum and eigenfunctions, localized in the small neighborhood of a periodic orbit, is developed in the presence of a homogeneous magnetic field and arbitrary scalar potential. The isolated periodic orbit is confined between two interfaces which could be planar, concave or even convex. Such a system represents a quantum electronic resonator, an analog of the well-known high-frequency optical or acoustic resonator with eigenmodes called 'bouncing ball vibrations'. The first step in the asymptotic analysis involves constructing a solitary localized asymptotic solution to the Schrödinger equation (electronic Gaussian beam—wavepackage). Then, the stability of a closed continuous family of periodic trajectories confined between two reflecting surfaces of the resonator boundary was studied. The asymptotics of the eigenfunctions were constructed as a superposition of two electronic Gaussian beams propagating in opposite directions between two reflecting points of the periodic orbits. The asymptotics of the energy spectrum are obtained by the generalized Bohr–Sommerfeld quantization condition derived as a requirement for the eigenfunction asymptotics to be periodic. For one class of periodic orbits, localized eigenstates were computed numerically by the finite element method using FEMLAB and proved to be in a very good agreement with those computed semiclassically.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Historically, the first attempts to develop a semiclassical analysis for high-energy eigenstates in the multidimensional case go back to the late 1950s when two famous papers by J B Keller were published [1, 2]. Then, in the late 1960s and 1970s, important results for high-energy asymptotics for elliptic boundary value problems were obtained by Babich and Buldyrev (see [3], 'whispering galleries and bouncing balls' asymptotics). Simultaneously, Maslov's canonical operator method was developed to construct semiclassical asymptotics for elliptic PDEs and quantum mechanics equations ([4]). A semiclassical approach to eigenstates associated with unstable periodic orbits (PO) for 'billiard' problems with Helmholtz and Schrödinger operators has been developed extensively for the last three decades. This was the result of the study of the correspondence between classical chaotic dynamics and its quantum analog, see for example [5]. A significant advance was recently achieved by Vergini and coauthors [6–8] in constructing the high-energy localized asymptotic solutions of chaotic eigenstates. These solutions are associated with unstable POs with hyperbolic structure, the so-called resonances, which helped to explain the appearance of 'scarring' in chaotic eigenstates.

Further development of the semiclassical analysis of this type of 'billiard' problems is very important. It is effectively applied in various fields of modern physics such as quantum information, nanoscience, electronic transport in semiconductors and many others. One of the examples of the application is quantum electronic transport through the system waveguide–resonator–waveguide (WRW). Here, resonant peaks of conductance are associated with certain unstable POs of the resonator (see [9–11]). The case is of particular interest when families of POs are controlled by an external magnetic field [12].

An other example is the conductance of a 'double barrier' structure with a potential well similar to that described in [13]. If the perfectly reflecting interfaces of the electronic resonator are replaced by a finite height and width potential barriers, the semiclassical asymptotic approach may be used in this case to solve the problem of conductance.

It is of particular interest to study, in the semiclassical approximation, the role of stable and unstable POs in electronic transport in the WRW system. This problem has attracted much interest in recent years, mainly because of the study of quantum interference in low-dimensional structures in nanoscience, e.g., in semiconductor quantum wells and in thin metal overlayers. One of the examples is a study of confinement of surface electron states in quantum resonators similar to the Fabry–Perot interferometer (see [14, 15]). Furthermore, effective methods have been developed to compute the probability density distribution of electrons in semiconductor heterostructures [16, 17] and the electron surface states of metals [18–20]. The WRW system has also been studied by many physicists, for instance, as the problem of the Fabry–Perot interferometer. Specific examples of WRW system include an electronic waveguide with embedded resonator [21] or a coherent electron waveguide with the resonant cavity formed between the two nanotube–electrode interfaces [22].

In this paper, we study the problem of the construction of the high-energy eigenstates of an electron inside a closed resonator. These states are localized near a stable PO in the presence of a homogeneous magnetic field and arbitrary scalar potential. The isolated periodic orbit is confined between two perfectly reflecting interfaces which could be planar, concave or even convex. Such a system represents a quantum electronic resonator which is an analog of the well-known high-frequency optical resonator, the so-called electromagnetic or acoustic 'bouncing balls vibrations'. Here, we concentrate more on the semiclassical analysis of the stability of POs and electronic eigenstates depending on the magnetic field. These results may be considered as the initial stage in developing a construction of resonances and 'scar functions' of unstable POs in the presence of a magnetic field.

The asymptotic analysis of the high-energy localized eigenstates presented here is similar to that used for optical resonators (see [3, 23]). As the first step in the analysis, we construct a solitary localized asymptotic solution in a neighborhood of the classical trajectory called an electronic Gaussian beam (wavepackage). In the second stage, we study the stability of a closed continuous family of trajectories in the asymptotic proximity to a PO confined between two reflecting interfaces. The stability analysis is based on the classical theory of linear Hamiltonian systems with periodic coefficients (the monodromy matrix analysis). The asymptotic eigenfunctions are constructed as a superposition of two modes—two electronic Gaussian beams propagating in opposite directions between two reflecting points of the periodic orbit. The asymptotic energy spectral series is given by the generalized Bohr–Sommerfeld quantization condition (see [1, 3, 4]). It is obtained as a requirement of the uniqueness and periodicity of the asymptotic eigenfunctions.

By giving a description to the general form asymptotic solution, we refer our results to a special class of POs for a homogeneous magnetic field and a quadratic potential. The latter one as described in Datta [24] 'is often a good description of actual potential in many electronic waveguides'. The key point in the asymptotic analysis is the quantization of the continuous one-parameter family of POs. Such quantization takes place in the range of the POs stability with respect to the parameter defining the family of POs. For one subclass of periodic orbits, these localized eigenstates were tested against eigenvalues and eigenfunctions computed by the finite element method using FEMLAB. In the paper, we show that, for a few chosen energy eigenvalues and eigenfunctions, agreement between the numerical results and those computed semiclassically is very good. An example is given which was computed by FEMLAB. It shows the excitation of one of the localized eigenstates at the intersection of a rectangular resonator and a waveguide. The eigenstate is excited by a waveguide electronic traveling mode in the case of resonance. This fact demonstrates the importance of the high-energy localized eigenstates of the electronic resonator considered in this paper.

It is worth noting that the method of asymptotical quantization we developed is similar to the Maslov spectral semiclassical approach (complex germ method, see [25, 26]). The Maslov complex germ method was applied to partially integrable systems such as the motion of charged particles in external electromagnetic fields with axial symmetry. However, we believe that the techniques our method is based on, namely, the boundary-layer asymptotic techniques, are more straightforward and less complicated. At the same time, there is a principal difference with respect to the asymptotic boundary-layer approach developed in [3, 23]). In the latter case, the construction of the high-energy localized eigenstates takes place for a single fixed stable PO independent of eigenfrequency. In our case, it is a quantization of a family of stable and unstable POs dependent on eigenenergy.

The paper is organized as follows. First, in section 2, we give a description of the families of PO which are being used in the application of the proposed semiclassical analysis. In section 3, the details of the boundary-layer semiclassical method used to construct the asymptotic solution of the Gaussian beam in the presence of a magnetic field and a scalar potential are presented. Subsequently, in section 4, the construction of high-energy localized eigenfunctions and PO stability analysis is discussed. Finally, in section 5, the generalized Bohr–Sommerfeld quantization condition leads to asymptotic formulae of the high-energy spectral series. For a few localized eigenmodes numerical tests are shown in figures for a special class of POs in the case of a homogeneous magnetic field and a quadratic potential.

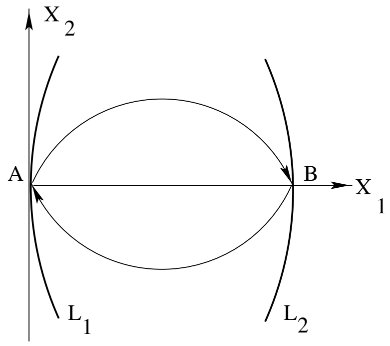

Figure 1. A periodic orbit inside the electronic resonator with a magnetic field and a quadratic potential.

2. Class of POs of an electronic resonator in a magnetic field and a quadratic potential

Consider a spectral problem for the Schrödinger operator describing an electron in the presence of a homogeneous magnetic field and arbitrary scalar potential:

$$\frac{1}{2m}\left\{\left(\hat{p}_1+\frac{\alpha x_2}{2}\right)^2+\left(\hat{p}_2-\frac{\alpha x_1}{2}\right)^2\right\}\psi+u(\mathbf{x})\psi=E\psi,$$
$$\mathbf{x}=(x_1,x_2),\qquad \hat{p}_1=\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_1},\qquad \hat{p}_2=\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_2},\qquad \alpha=\frac{eB}{c},\tag{1}$$

with the magnetic potential in axial gauge $\mathbf{A}=B/2(-x_2,x_1,0)$. Here m, e are the mass and charge of a particle, c is the speed of light and $\hbar$ is the Plank constant which is a small parameter ($\hbar\to 0$). We study high-energy spectral problem in the semiclassical approximation in a domain confined between two concave, convex or flat reflecting interfaces $L_{1,2}$ (see figure 1). The wavefunction satisfies Dirichlet boundary condition on the interfaces $L_{1,2}$

$$\psi|_{L_{1,2}}=0.$$

In general case, if high-energy localized eigenstates are sought, which decay exponentially away from the resonator axis AB, the separation of variables will not help construct exact solution due to the difficulty of satisfying the boundary conditions.

Consider a class of continuous families of POs which are symmetric with respect to both axis, with two reflection points A, B. In the case of quadratic potential $u(\mathbf{x})=\beta x_2^2/2$, the equations describing PO as solutions of the corresponding Hamilton system are easily obtained and given by

$$x_1=f_1(t,\pi_1,\pi_2)=\frac{\pi_1}{m}\left(1-\frac{\omega^2}{\Omega^2}\right)t+\frac{\pi_2\omega}{m\Omega^2}(1-\cos\Omega t)+\frac{\pi_1\omega^2}{m\Omega^3}\sin\Omega t,$$
$$x_2=f_2(t,\pi_1,\pi_2)=\frac{\pi_1\omega}{m\Omega^2}(\cos\Omega t-1)+\frac{\pi_2}{m\Omega}\sin\Omega t\tag{2}$$

for the upper part $0<t<t_0$, and

$$x_1=D-\frac{\pi_1}{m}\left(1-\frac{\omega^2}{\Omega^2}\right)t-\frac{\pi_2\omega}{m\Omega^2}(1-\cos\Omega t)-\frac{\pi_1\omega^2}{m\Omega^3}\sin\Omega t,$$
$$x_2=-\frac{\pi_1\omega}{m\Omega^2}(\cos\Omega t-1)-\frac{\pi_2}{m\Omega}\sin\Omega t\tag{3}$$

for the lower part $t_0 < s < 2t_0$, where $D = |AB|$ is the width of the resonator, π_1 and π_2 are the values of the components of momentum p_1 and p_2 at the point $A, \omega = \alpha/m$ is the cyclotronic frequency and

$$\Omega = \sqrt{\omega^2 + \frac{\beta}{m}}.$$

In the system of equations (2) and (3) π_1 is a fixed parameter whereas π_2 and t_0 as functions of π_1 are determined uniquely by the equations

$$f_1(t_0, \pi_1, \pi_2) = D, \qquad f_2(t_0, \pi_1, \pi_2) = 0.$$

Thus, we have the continuous family of POs with respect to parameter π_1. The classical energy for PO is

$$E = \frac{1}{2m}\left(\pi_1^2 + \pi_2^2\right). \tag{4}$$

In the limiting case $\beta = 0$, the family of POs permits transform into a family of arcs of circle with cyclotronic radius r_ω of electron in a constant magnetic field

$$x_1 = r_\omega\left[\sin\left(\frac{s}{r_\omega} - \gamma\right) + \sin(\gamma)\right], \qquad x_2 = r_\omega\left[\cos\left(\frac{s}{r_\omega} - \gamma\right) - \cos(\gamma)\right]$$

for the upper arc $0 < s < s_0$ (s is the arc length) and

$$x_1 = D - r_\omega\left[\sin\left(\frac{s}{r_\omega} - \gamma\right) + \sin(\gamma)\right], \qquad x_2 = r_\omega\left[\cos(\gamma) - \cos\left(\frac{s}{r_\omega} - \gamma\right)\right] \tag{5}$$

for the lower arc $s_0 < s < 2s_0$. Here, γ is the incidence angle of PO at the reflecting points A, B, and measured from the axis X_1. In this case, we have

$$r_\omega = \frac{1}{\omega}\sqrt{\frac{2E}{m}}, \qquad D = 2r_\omega \sin(\gamma), \qquad s_0 = 2\gamma r_\omega,$$

and the total length of PO is $2s_0$. Thus, we have the continuous family of POs with respect to parameter γ, and

$$\pi_1 = \frac{m\omega D}{2}\cot\gamma, \qquad \pi_2 = \frac{m\omega D}{2}.$$

In the opposite limiting case when $\alpha = 0$, the family of POs transforms into the horizontal segment AB. This is the case where the separation of variables gives exact high-energy localized eigenstates (see section 5) if the resonator's interfaces are flat.

By introducing new variables $\sigma_{1,2}, \gamma_{1,2}$ and the dimensionless energy e by means of the following formulae

$$\sigma_1 = \omega/\omega_0, \qquad \beta = m\omega_0^2\sigma_2, \qquad \pi_i = D\omega_0 m\gamma_i, \qquad i = 1, 2, \qquad E = D^2\omega_0^2 me,$$

with ω_0 as the reference frequency, we obtain a dimensionless form of the problem with the Hamiltonian (1). Now, the continuous family of POs is determined by γ_1. The small dimensionless asymptotic parameter becomes

$$\epsilon = \frac{\hbar}{D^2 m\omega_0}.$$

Then, we have

$$\frac{1}{2}\left(-\Delta - \frac{\mathrm{i}\sigma_1}{\epsilon}x_2\frac{\partial}{\partial x_1} + \frac{\mathrm{i}\sigma_1}{\epsilon}x_1\frac{\partial}{\partial x_2} + \frac{\sigma_1^2\left(x_1^2 + x_2^2\right)}{4\epsilon^2}\right)\psi + \frac{\sigma_2 x_2^2}{2\epsilon^2}\psi = \epsilon^{-2}e\psi.$$

The classical energy for PO is

$$e = \tfrac{1}{2}\left(\gamma_1^2 + \gamma_2^2\right). \tag{6}$$

Assuming that $D = 1$, for the case $\sigma_1 = 0.5, \sigma_2 = 1$, two types of POs are shown in figure 2: (*a*) $\gamma_1 = 1, t_0 = 1.093$ and (*b*) $\gamma_1 = 1.5, t_0 = 3.5228$.

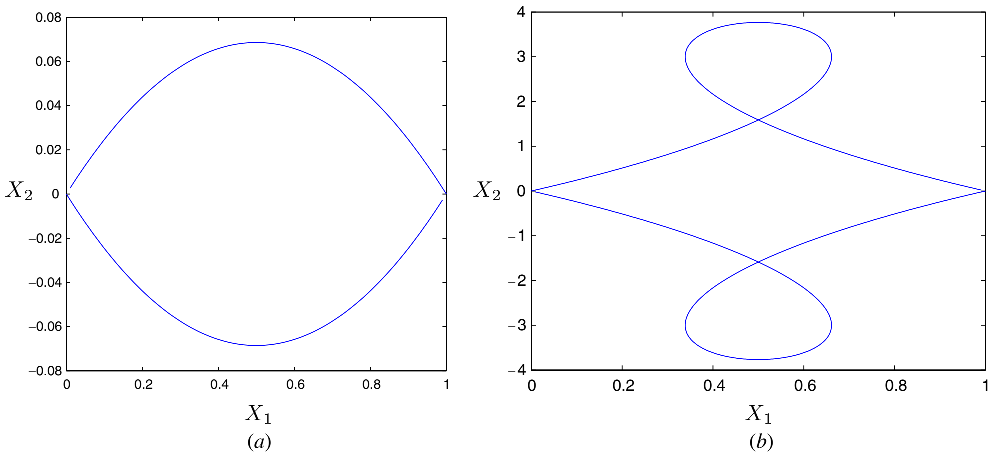

Figure 2. Two types of PO.

3. Electronic Gaussian beams

Let $\mathbf{x}_0 = \left(x_1^{(0)}(s), x_2^{(0)}(s)\right)$ be a particle classical trajectory, where s is the arc length measured along a trajectory. Consider the neighborhood of the trajectory in terms of local coordinates s, n, where s is the arc length measured along the trajectory and n is the distance along vector normal to the trajectory, such that

$$\mathbf{x} = \mathbf{x}_0(s) + \mathbf{e_n}(s)n,$$

where $\mathbf{e_n}(s)$ is the unit normal vector to the trajectory.

Following [23], we apply the asymptotic boundary-layer method to the Schrödinger equation (1). We assume that the width of the boundary layer is determined by $|n, \dot{n}| = O(\sqrt{\hbar})$ as $\hbar \to 0$. Introducing $\nu = n/\sqrt{\hbar} = O(1)$, we seek asymptotic solution to (1) in the form

$$\begin{aligned}
\psi &= \frac{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S_0+S_1 n)}}{\sqrt{a(s)}} \sum_{j=0}^{+\infty} \psi_j(s,\nu)\hbar^{j/2}, \\
S_0(s) &= \int \left(a(s) - \frac{\alpha}{2}\left(x_1^{(0)}\gamma_1 + x_2^{(0)}\gamma_2\right)\right) \mathrm{d}s, \\
S_1(s) &= \frac{\alpha}{2}\left(x_1^{(0)}\gamma_2 - x_2^{(0)}\gamma_1\right), \\
a(s) &= \sqrt{2m(E - u_0(s))},
\end{aligned} \tag{7}$$

where

$$u(\mathbf{x}) = u_0(s) + u_1(s)n + u_2(s)n^2 + \cdots,$$
$$\gamma_i(s) = \langle \mathbf{e_n}(s), \mathbf{e}_i \rangle, \qquad i = 1, 2,$$

where symbol $\langle\, ,\, \rangle$ means the scalar product. Thus, for unknown $\psi_j(s,\nu)$ we obtain a recurrent system

$$L_0\psi_0 = 0, \qquad L_0\psi_1 + L_1\psi_0 = 0, \ldots$$

with differential operators $L_0, L_1, \ldots,$ such that the operator L_0 describes the boundary-layer Schrödinger-type equation (see the appendix)

$$L_0\psi_0 = \frac{\partial^2}{\partial \nu^2}\psi_0 + 2\mathrm{i}a\frac{\partial}{\partial s}\psi_0 - \nu^2 a^2 d\psi_0, \tag{8}$$

where

$$d(s) = \frac{u_2}{E-u_0} + \frac{u_1^2}{4(E-u_0)^2} - \frac{u_1}{\rho(E-u_0)} - \frac{\alpha}{\rho a},$$

and $\rho(s)$ is the radius of curvature of the trajectory. The equation $L_0\psi_0 = 0$ has a solution

$$\psi_0^{(0)}(\mathbf{x}, E) = \frac{\mathrm{e}^{\mathrm{i}\Gamma\nu^2/2}}{\sqrt{z}}, \qquad \Gamma = a\frac{\dot z}{z}, \tag{9}$$

where $\dot z$ means the derivative with respect to s (see [3, 23]). Here, Γ satisfies the Ricatti equation,

$$\dot\Gamma + \frac{1}{a}\Gamma^2 + ad = 0. \tag{10}$$

The function $z(s)$ satisfies the system of equations in variations in the Hamiltonian form

$$\dot z = p/a(s), \qquad \dot p = -a(s)\,\mathrm{d}(s)z \tag{11}$$

with the Hamiltonian

$$H(z,p) = \frac{p^2}{2a(s)} + \frac{a(s)\,\mathrm{d}(s)z^2}{2}.$$

The crucial point of the analysis is that it is possible to choose a solution of (11) in such a way that $\mathrm{Im}\,\Gamma > 0$, thus providing asymptotic localization of ψ, namely, if $z(s)$ is a complex solution to (11). Wronskian of $\mathrm{Re}(z(s))$ and $\mathrm{Im}(z(s))$ may be chosen in such a way that

$$a(s)W(\mathrm{Re}(z), \mathrm{Im}(z)) = \tfrac{1}{2}.$$

Then, the following inequality holds:

$$\mathrm{Im}(\Gamma(s)) = \frac{a(s)W(\mathrm{Re}(z), \mathrm{Im}(z))(s)}{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} = \frac{1}{2|z(s)|^2} > 0$$

along the trajectory. It gives the localization. We assume that there are no turning points along the trajectory such that $a(s) > 0$ for all s.

Similarly (see [23, 3]), we introduce the annihilation and creation operators

$$\Lambda(E) = z\frac{\partial}{\mathrm{i}\partial\nu} - \dot z a\nu, \qquad \Lambda^*(E) = \bar z\frac{\partial}{\mathrm{i}\partial\nu} - \dot{\bar z}a\nu, \tag{12}$$

for which the following commutator relations hold:

$$[L_0, \Lambda] = 0, \qquad [L_0, \Lambda^*] = 0, \qquad [\Lambda, \Lambda^*] = 1, \qquad [\Lambda, \Lambda^{*m}] = m\Lambda^{*(m-1)},$$

for $m = 2, 3, \ldots$, and

$$\Lambda\psi_0^{(0)} = 0 \qquad \text{if} \qquad L_0\psi_0^{(0)} = 0.$$

Thus, we obtain a countable set of solutions to $L_0\psi_0 = 0$ in the form

$$\psi_0^{(m)}(\mathbf{x}, E) = \Lambda^{*m}(E)\psi_0^{(0)}(\mathbf{x}, E). \tag{13}$$

All these solutions are linear independent and orthogonal in the sense of the scalar product

$$\int_{-\infty}^{+\infty} \psi_0^{(m_1)}(s,\nu)\bar\psi_0^{(m_2)}(s,\nu)\,\mathrm{d}\nu = \delta_{m_1 m_2} m_1!\sqrt{\frac{2\pi}{a(s)}}.$$

The solution $\psi_0^{(m)}$ may be written as

$$\psi_0^{(m)}(\mathbf{x}, E) = Q_m(z,\nu)\frac{\mathrm{e}^{\mathrm{i}\Gamma\nu^2/2}}{\sqrt{z(s)}},$$

where $Q_m(z,\nu)$ are up to a constant the Hermitian polynomials with respect to ν.

4. Construction of eigenfunctions and PO stability analysis

We assume that we deal with a continuous family of POs symmetric with respect to both axis. The trajectory of PO consists of two symmetric parts between two reflection points A and B (see figure 1). We seek the asymptotic solution of the eigenfunction localized in the neighborhood of the PO as a combination of two electronic Gaussian beams

$$\psi(\mathbf{x}, E) = \psi_1(\mathbf{x}, E) + R\psi_2(\mathbf{x}, E),$$

described by

$$\psi_{1,2}(\mathbf{x}, E) = \exp\left(\frac{\mathrm{i}}{\hbar}\left(S_0(s) + S_1(s)n + \frac{1}{2}\frac{p_{1,2}(s)}{z_{1,2}(s)}n^2\right)\right)$$
$$\frac{Q_m(z_{1,2}(s), \nu)}{\sqrt{a(s)z_{1,2}(s)}}(1 + O(\hbar^{1/2})),$$

where each beam is related with the corresponding part of the periodic orbit, namely, ψ_1 is determined by $z_1(s)$, $p_1(s)$ for $0 < s < s_0$ and propagating along the upper part of the orbit, whereas ψ_2 is determined by $z_2(s)$, $p_2(s)$ for $s_0 < s < 2s_0$ and propagating down along the lower part of the orbit. For the reflection coefficient R in the case of Dirichlet boundary condition we have $R = -\mathrm{i}$.

Assume that we have PO and a quasi-periodic Floquet solution of the Hamiltonian system (11) satisfies

$$\begin{pmatrix} z(s+2s_0) \\ p(s+2s_0) \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} z(s) \\ p(s) \end{pmatrix} = \lambda \begin{pmatrix} z(s) \\ p(s) \end{pmatrix}, \tag{14}$$

where M is the monodromy matrix, mapping for a period $2s_0$, and λ is the Floquet multiplier. On our way to construct asymptotics of the periodic solution $\psi = \psi_1 + R\psi_2$, first we must determine a quasi-periodic bounded Floquet solution $(z(s), p(s))$ of the Hamiltonian system (11) in the complex phase space $C^2_{z,p}$. The structure of the monodromy matrix M is given by the following product:

$$M = M_2 R^A M_1 R^B, \qquad \det M = 1,$$

where M_1 and M_2 are fundamental matrices of the system (11) describing the evolution $(z(s), p(s))$ for $0 < s < s_0$ and $s_0 < s < 2s_0$, correspondingly. While, R^A and R^B are reflection matrices at points A and B (see figure 1). For instance, as a result of reflection at the point B, we obtain that

$$\begin{pmatrix} z_2(s_0) \\ p_2(s_0) \end{pmatrix} = R^B \begin{pmatrix} z_1(s_0) \\ p_1(s_0) \end{pmatrix}, \qquad R^B = \begin{pmatrix} -1 & 0 \\ R^B_{21} & -1 \end{pmatrix}, \qquad R^B_{21} = 2\frac{\sqrt{2m(E - u_0(B))}}{\rho_B \cos\gamma}, \tag{15}$$

where γ is the angle of incidence of the trajectory at the point B, ρ_B is the radius of curvature of the reflecting interface at the point B. For R^A we obtain similar result. The formulae for R^A and R^B were derived by requiring the continuity condition of the phase function S of the incident and reflected beams along the reflecting interface (see [3, 23, 27, 28]). Below, when discussing the numerical results, we assume that the resonator's interfaces $L_{1,2}$ are flat. As a result, we obtain

$$R^A = R^B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $M = M_2 M_1$. In general case, the entries of $M_{1,2}$ are to be determined numerically as the Hamiltonian system (11) has variable coefficients.

The classical theory of linear Hamiltonian systems with periodic coefficients states that, if $|\mathrm{Tr}\, M| < 2$, we have a stable PO (elliptic fixed point, for example, see [27]) and there exist

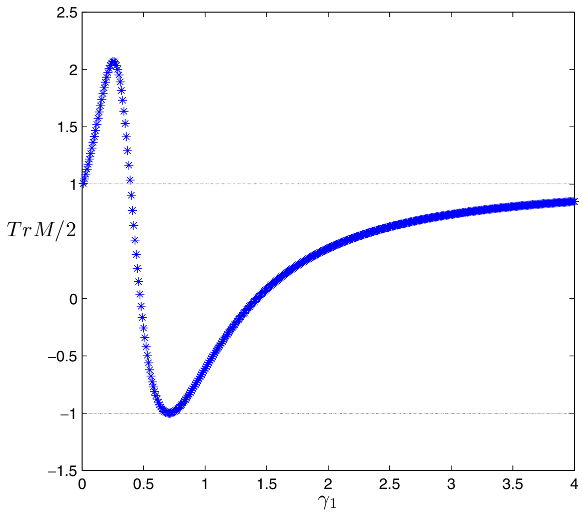

Figure 3. Dependence of $\mathrm{Tr}\, M/2$ on γ_1.

two bounded Floquet's solutions for $-\infty < s < +\infty$, namely, $(z(s), p(s))$ and $(\bar{z}(s), \bar{p}(s))$ with Floquet's multipliers $\lambda_{1,2} = \mathrm{e}^{\pm \mathrm{i}\varphi}$ $(0 < \varphi < \pi)$, which are solutions of

$$\lambda^2 - \mathrm{Tr}\, M\lambda + 1 = 0.$$

We choose the solution for which the inequality $W(\mathrm{Re}(z), \mathrm{Im}(z))(0) > 0$ holds and, consequently, we obtain the localization of the Gaussian beam along the entire PO as

$$\mathrm{Im}\left(\frac{p(s)}{z(s)}\right) > 0, \qquad -\infty < s < +\infty.$$

In the case $|\mathrm{Tr}\, M| > 2$, PO is unstable (hyperbolic dynamics), and this case is not discussed in the paper. In the case $|\mathrm{Tr}\, M| = 2$, stability takes place if there is no adjoint vector of the monodromy matrix M. In figure 3, we show the dependence of $\mathrm{Tr}\, M/2$ on γ_1 for $\sigma_1 = 0.5, \sigma_2 = 1$. In this case, we observe a stability range $0.39\ldots < \gamma_1$. This depends on the magnetic field and potential (parameters α, β).

If the potential is zero ($\beta = 0$), the entries of $M_{1,2}$ are computed analytically, and we have (see formulae (5))

$$M_1 = M_2 = \begin{pmatrix} \cos\left(\frac{s_0}{r_\omega}\right) & \frac{r_\omega}{\sqrt{2mE}} \sin\left(\frac{s_0}{r_\omega}\right) \\ -\frac{\sqrt{2mE}}{r_\omega} \cos\left(\frac{s_0}{r_\omega}\right) & \cos\left(\frac{s_0}{r_w}\right) \end{pmatrix}.$$

It is important to note that in this case every PO is stable as $|\mathrm{Tr}\, M(\gamma)| < 2$ for all γ $(0 < \gamma < \pi/2)$. In the opposite limiting case $\alpha = 0$, the family of POs is the segment AB, and the monodromy matrix is given by

$$M = \begin{pmatrix} \cos\left(2D\sqrt{\frac{\beta}{2E}}\right) & -\sqrt{\frac{\beta}{2E}} \sin\left(2D\sqrt{\frac{\beta}{2E}}\right) \\ \sqrt{\frac{2E}{\beta}} \sin\left(2D\sqrt{\frac{\beta}{2E}}\right) & \cos\left(2D\sqrt{\frac{\beta}{2E}}\right) \end{pmatrix}.$$

It is clear that in this case the orbit–segment AB is always stable as $|\mathrm{Tr}M| \leqslant 2$. If $\mathrm{Tr}\,M = \pm 2$, then

$$M = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

5. Quantization condition and numerical results

Below we assume that we have a range of parameter (π_1 or γ_1) defining stable continuous families of POs. The requirement that the constructed asymptotic solution of the eigenfunction $\psi = \psi_1 + R\psi_2$ is periodic with period $2s_0$ leads to the generalized Bohr–Sommerfeld quantization condition determining semiclassical asymptotics of the high-energy spectral series. Namely, after the integration around the closed loop of PO, the total variation of the classical action S and the amplitude of ψ must be equal to $2\pi m_1$. Thus, we obtain

$$\int_0^{2s_0} \sqrt{2m(E - u(\mathbf{x}_0(s)))}\,\mathrm{d}s + \alpha A = \hbar(2\pi m_1 + (m_2 + 1/2)\varphi), \tag{16}$$

where $m_{1,2} \in Z$ are the longitudinal and the transversal quantization indices. The index m_2 appears due to the variation of $\psi_0^{(m)}(\mathbf{x}, E)$ (see formulae (12) and (13)). Here

$$A = -\frac{\alpha}{2}\int_0^{2s_0} \left(\left(x_1^{(0)}\gamma_1 + x_2^{(0)}\gamma_2\right)\right)\mathrm{d}s$$

is the area encircled by the periodic orbit. If the potential $u(\mathbf{x}) = 0$, the quantization condition (16) may be simplified as follows:

$$\frac{2\gamma + \sin 2\gamma}{4\sin^2\gamma} = \frac{\hbar}{m\omega D^2}(2\pi m_1 + (m_2 + 1/2)\varphi).$$

Having a continuous family of periodic orbits depending on E or π_1, the quantization condition is satisfied only for a discrete set of energy levels $E = E_{m_1,m_2}$. It is clear that the quantization condition may be fulfilled only if the longitudinal index m_1 is large as $\hbar \to 0$. At the same time, the transversal index $m_2 = 0, 1, 2, \ldots$ should be of the order 1 as the larger values of m_2 would lead to the asymptotic solution $\psi = \psi_1 + R\psi_2$ not being localized. Thus, we obtain

$$E_{m_1,m_2} = E_{m_1}^{(0)} + \hbar E_{m_1,m_2}^{(1)} + O(\hbar^2).$$

Now, the principal term $E_{m_1}^{(0)}$ is to be found from

$$\int_0^{2s_0} \sqrt{2m\left(E_{m_1}^{(0)} - u(\mathbf{x}_0(s))\right)}\,\mathrm{d}s + \alpha A = 2\pi m_1\hbar.$$

Then, the next order term of the energy levels are obtained by applying standard perturbation scheme

$$E_{m_1,m_2}^{(1)} = \frac{(m_2 + 1/2)\varphi}{\Phi'\left(E_{m_1}^{(0)}\right)},$$

where

$$\Phi(E) = \int_0^{2s_0} \sqrt{2m(E - u(\mathbf{x}_0(s)))}\,\mathrm{d}s + \alpha A(E).$$

If the potential $u(\mathbf{x}) = 0$, we have

$$\Phi(E) = m\omega D^2 \frac{2\gamma + \sin 2\gamma}{4\sin^2\gamma}.$$

It is worth noting that we first perform the quantization of the parameter defining the stable families of POs (π_1 or γ_1), and then we obtain the corresponding quantized values of the energy by means of formulae (4). After the energy eigen-level was determined from the quantization condition (16), we obtain the asymptotic expansion

$$\psi_{m_1,m_2}(\mathbf{x}) = \psi_1(\mathbf{x}, E_{m_1,m_2}) + R\psi_2(\mathbf{x}, E_{m_1,m_2}),$$

where

$$\psi_0^{(m_2)}\left(s, \nu, E_{m_1}^{(0)}\right) = \left(\Lambda^*\left(E_{m_1}^{(0)}\right)\right)^{m_2} \psi_0^{(0)}\left(s, \nu, E_{m_1}^{(0)}\right)$$
$$= \exp\left(\frac{\mathrm{i}}{2}\frac{p_{1,2}(s)}{z_{1,2}(s)}\nu^2\right)\frac{Q_{m_2}(z_{1,2}(s), \nu)}{\sqrt{z_{1,2}(s)}},$$

whereas $S_0(s)$ and $S_1(s)$ are to be computed for E_{m_1,m_2}.

It is important to note that the high-energy spectral series of the eigenvalues and the localized eigenfunctions for the families of stable POs may be constructed not in the first approximation only. Higher order terms could be obtained using techniques of the perturbation theory similar to [23].

It was mentioned before that in the case of zero magnetic field and flat reflecting interfaces $L_{1,2}$ the problem

$$\frac{1}{2m}\left(\hat{p}_1^2 + \hat{p}_2^2\right)\psi + \beta\frac{x_2^2}{2}\psi = E\psi, \qquad \psi|_{L_{1,2}} = 0 \tag{17}$$

is exactly solvable by the separation of variables. Thus, we obtain the following solution for the eigenfunctions:

$$\psi_{m_1,m_2} = \text{const}\sin\left(\frac{p_1 x_1}{\hbar}\right)\Phi_{m_2}(x_2), \tag{18}$$

where

$$\Phi_{m_2}(x_2) = \left(\frac{\sqrt{m\beta}}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\sqrt{m\beta}}{2\hbar}x_2^2\right)H_{m_2}\left(\sqrt{\frac{\sqrt{m\beta}}{\hbar}}x_2\right),$$

and the energy eigen-levels are

$$E_{m_1,m_2} = \frac{p_1^2}{2m} + \hbar\sqrt{\frac{\beta}{m}}(m_2 + 1/2), \qquad p_1 = \frac{\pi m_1\hbar}{D}, \tag{19}$$

where $H_{m_2}(x)$ are the Hermitian polynomials of the order m_2.

In the asymptotic approximation $m_1\hbar = O(1), m_2 = O(1)$ as $\hbar \to 0$, our semiclassical approach in the limit $\alpha = 0$ for the eigenfunction asymptotics gives the same formula (18) with the only difference for

$$p_1 = \sqrt{2mE} - \hbar\sqrt{\frac{\beta}{2E}}\left(m_2 + \frac{1}{2}\right), \tag{20}$$

where the energy satisfies the quantization condition

$$\sqrt{2mE} = \hbar\left(\frac{\pi m_1}{D} + \sqrt{\frac{\beta}{2E}}\left(m_2 + \frac{1}{2}\right)\right). \tag{21}$$

From equations (20), (21) to the leading order we obtain that

$$p_1 = \frac{\pi m_1\hbar}{D}.$$

Taking into account this result, from (21) we find that the energy eigen-levels are given by formulae (19).

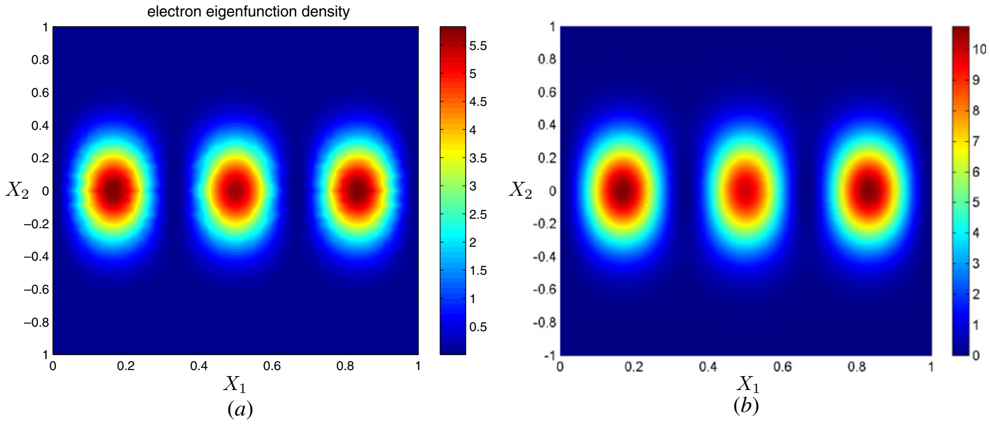

Figure 4. Electronic eigenfunction density $|\psi|^2$ for the state $m_1 = 3, m_2 = 0$ computed by semiclassical analysis for $e = 0.533$ (*a*) and by FEMLAB for $e = 0.506$ (*b*).

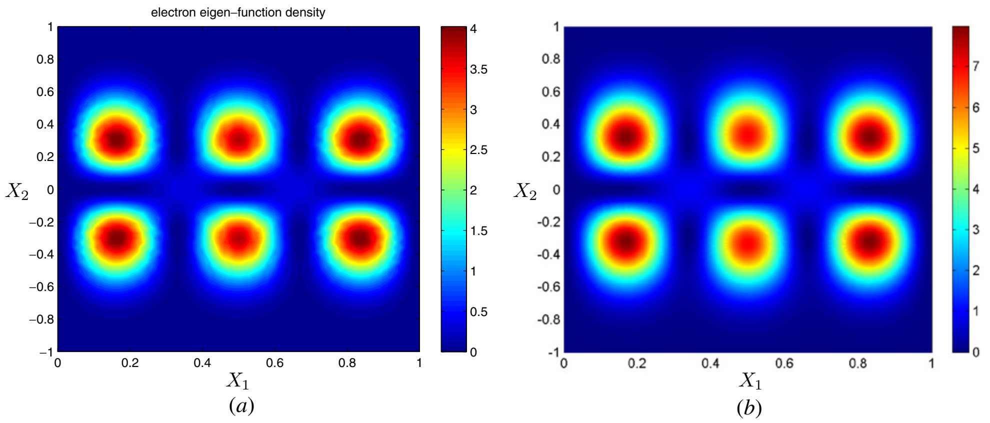

Figure 5. Electronic eigenfunction density $|\psi|^2$ for the state $m_1 = 3, m_2 = 1$ computed by semiclassical analysis for $e = 0.634$ (*a*) and by FEMLAB for $e = 0.607$ (*b*).

For the subclass of POs considered in the paper with $\sigma_1 = 0.5, \sigma_2 = 1, \epsilon = 0.1$, and the flat resonator's interfaces $L_{1,2}$, the high-energy localized eigenstates were tested against the eigenvalues and the eigenfunctions computed by the finite element method using FEMLAB. In figures 4–6, it is shown that for the states $m_1 = 3, m_2 = 0, 1, 2$ the agreement between the numerical results for the energy eigen-levels and eigenfunction density $|\psi|^2$ and those computed semiclassically is very good. In figure 7, we plotted the eigenfunction density $|\psi|^2$ computed semiclassically for the states $m_1 = 13, m_2 = 0, 1$ in the case of $\sigma_1 = 0.5, \sigma_2 = 1, \epsilon = 0.02$ and flat interfaces.

In figure 8, an example is shown which was computed by FEMLAB. It shows the excitation of the high-energy localized eigenstate $m_1 = 3, m_2 = 2$ mentioned above (see figure 6) with $\sigma_1 = 0.5, \sigma_2 = 1, \epsilon = 0.1$ in the intersection of the rectangular resonator and the horizontal waveguide. Dirichlet boundary condition was imposed along the boundary of the WRW system except for the radiation conditions on the left and right ends of the waveguide. The fact that the eigenstate is excited by waveguide electronic traveling mode in the case

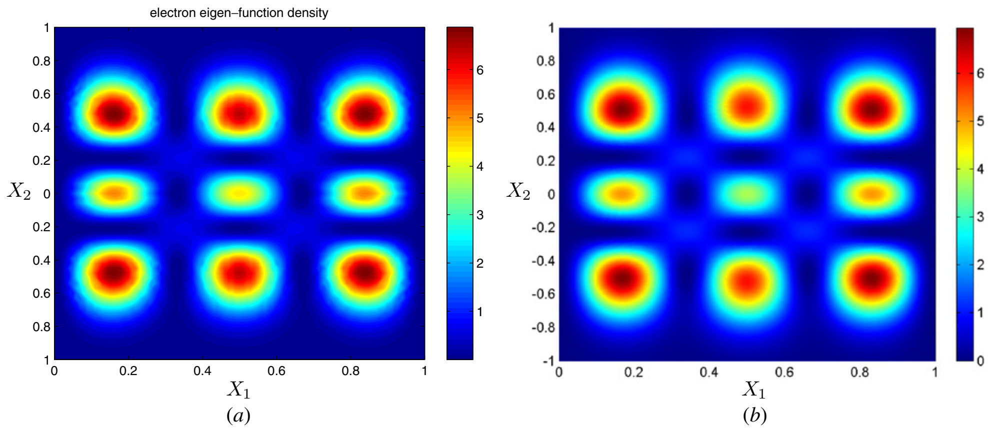

Figure 6. Electronic eigenfunction density $|\psi|^2$ for the state $m_1 = 3, m_2 = 2$ computed by semiclassical analysis for $e = 0.728$ (*a*) and by FEMLAB for $e = 0.709$ (*b*).

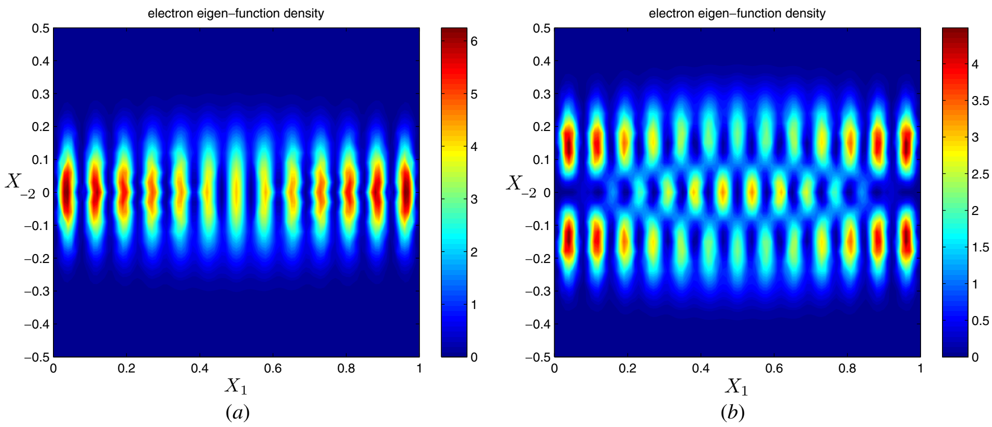

Figure 7. Electronic eigenfunction density $|\psi|^2$ computed by semiclassical analysis for the state $m_1 = 13, m_2 = 0$ with $\epsilon = 0.381$ (*a*) and $m_1 = 13, m_2 = 1$ with $e = 0.4024$ (*b*), $\epsilon = 0.02$.

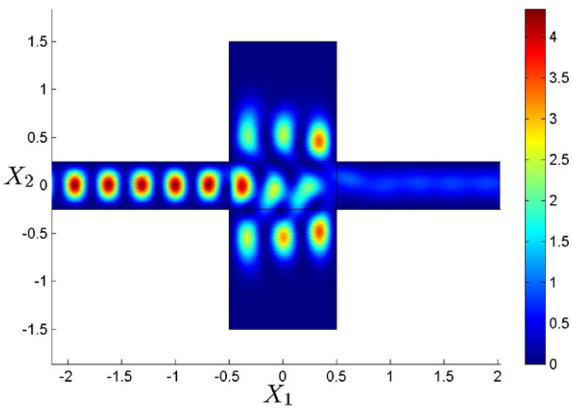

Figure 8. The excitation of the state $m_1 = 3, m_2 = 2$ in the resonator by traveling resonance mode inside waveguide.

of resonance demonstrates the importance of the high-energy localized eigenstates of the electronic resonator considered in the paper.

The semiclassical analysis of the high-energy localized spectral series for the PO shown in figure 2(*b*) and POs with more complicated structure will be considered in the future publications.

6. Conclusion

For the electronic resonator in a magnetic and arbitrary electric field the semiclassical approximation was applied to construct the asymptotic spectral series of high-energy eigenlevels. Corresponding eigenfunctions are localized in the neighborhood of classical stable periodic orbits similar to 'bouncing balls' high-frequency vibrations of optical or acoustic resonators. Their asymptotic expansions are obtained as superposition of Gaussian beams (Gaussian wavepackages), incident and reflected between the resonator interfaces. The asymptotics of the energy spectral series are derived from the quantization condition of the generalized Bohr–Sommerfeld type. For one class of periodic orbits localized eigenstates were computed numerically by the finite element method using FEMLAB and proved to be in a very good agreement with those computed semiclassically. To illustrate the behavior of localized eigenfunctions of the electronic resonator a couple of portraits of densities of the electron eigenfunctions was plotted for two high-energy eigenvalues determined by the quantization condition. An example was shown, which was computed by FEMLAB, demonstrating that these high-energy eigenstates could be excited at the intersections of a waveguide and a resonator. We hope that the current model of high-energy localized eigenstates of the electronic resonator could be generalized to study corresponding transport problems of 2D electronic gas in the system waveguide–resonator–waveguide, and it may be developed for the resonances of unstable periodic orbits of electronic resonators in the presence of a magnetic field.

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Appendix

In the appendix, the basic steps of derivation of the boundary-layer Schrödinger-type equation (8) are outlined. The initial Schrödinger equation (1) may be written in the form

$$\Delta\psi + \mathrm{i}\alpha\hbar^{-1}\left(x_2\psi_{x_1} - x_1\psi_{x_2}\right) + \hbar^{-2}\left(2m(E - u(\mathbf{x})) - \frac{\alpha^2}{4}\left(x_1^2 + x_2^2\right)\right)\psi = 0.$$

Using the local coordinates s, n, related with the fixed trajectory $x_1^{(0)}(s), x_2^{(0)}(s)$, we have

$$x_1 = x_1^{(0)}(s) + n\gamma_1(s), \qquad x_2 = x_2^{(0)}(s) + n\gamma_2(s),$$

$$\psi_{x_1} = \left(1 - \frac{n}{\rho}\right)^{-1}\left(\psi_s\gamma_2 + \psi_n\gamma_1\left(1 - \frac{n}{\rho}\right)\right),$$

$$\psi_{x_2} = \left(1 - \frac{n}{\rho}\right)^{-1}\left(\psi_n\gamma_2\left(1 - \frac{n}{\rho}\right) - \psi_s\gamma_1\right),$$

$$2m(E - u(\mathbf{x})) = a^2(s) + a_1 n + a_2 n^2 + \cdots, \qquad a_1 = -2mu_1, \qquad a_2 = -2mu_2.$$

The Schrödinger equation in terms of s, n coordinates becomes

$$\left(1-\frac{n}{\rho}\right)^{-1}\frac{\partial}{\partial n}\left(\left(1-\frac{n}{\rho}\right)\psi_n\right)+\left(1-\frac{n}{\rho}\right)^{-1}\frac{\partial}{\partial s}\left(\left(1-\frac{n}{\rho}\right)^{-1}\psi_s\right)$$
$$+\frac{\mathrm{i}\alpha\hbar^{-1}}{1-\frac{n}{\rho}}\left(\left(x_2^{(0)}+n\gamma_2\right)\left(\psi_s\gamma_2+\psi_n\gamma_1\left(1-\frac{n}{\rho}\right)\right)\right.$$
$$\left.-\left(x_1^{(0)}+n\gamma_1(s)\right)\left(\psi_n\gamma_2\left(1-\frac{n}{\rho}\right)-\psi_s\gamma_1\right)\right)$$
$$+\hbar^{-2}\left(a^2(s)+a_1n+a_2n^2-\frac{\alpha^2}{4}\left(\left(x_1^{(0)}+n\gamma_1(s)\right)^2+\left(x_2^{(0)}+n\gamma_2(s)\right)^2\right)\right)\psi+\cdots=0.$$

The width of the boundary layer is determined by $|n,\dot{n}|=O(\sqrt{\hbar})$ as $\hbar\to 0$. Here and below in the Schrödinger equation we neglect the terms of the order higher than $\hbar^{-1}$. Introducing

$$\psi=\exp\left(\frac{\mathrm{i}}{\hbar}(S_0(s)+S_1(s)n)\right)(U(s,n)+O(\hbar^{1/2})),$$

we obtain

$$-\mathrm{i}\frac{\hbar^{-1}}{\rho}S_1U+U_{nn}+\left(1-\frac{n}{\rho}\right)\left(-\hbar^{-2}S_1^2U+2\mathrm{i}\hbar^{-1}S_1U_n+\left(1+\frac{n}{\rho}+\frac{n^2}{\rho^2}\right)\right.$$
$$\times\left(-\hbar^{-2}(\dot{S}_0+\dot{S}_1n)^2U+\mathrm{i}\hbar^{-1}(\ddot{S}_0+\ddot{S}_1n)U+2\mathrm{i}\hbar^{-1}(\dot{S}_0+\dot{S}_1n)U_s\right)+\mathrm{i}\alpha\hbar^{-1}\left(x_2^{(0)}+n\gamma_2\right)$$
$$\times\left(U_s\gamma_2+U_n\gamma_1\left(1-\frac{n}{\rho}\right)+\mathrm{i}\hbar^{-1}\gamma_2(\dot{S}_0+\dot{S}_1n)U+\mathrm{i}\hbar^{-1}\gamma_1S_1U\left(1-\frac{n}{\rho}\right)\right)$$
$$-\mathrm{i}\alpha\hbar^{-1}\left(x_1^{(0)}+n\gamma_1\right)\left(\gamma_2(U_n+\mathrm{i}\hbar^{-1}S_1U)\left(1-\frac{n}{\rho}\right)-\gamma_1(\mathrm{i}\hbar^{-1}(\dot{S}_0+\dot{S}_1n)U+U_s)\right)$$
$$+\hbar^{-2}U\left(1-\frac{n}{\rho}\right)\left(a^2(s)+a_1n+a_2n^2-\frac{\alpha^2}{4}\left(\left(x_1^{(0)}\right)^2+\left(x_2^{(0)}\right)^2\right.\right.$$
$$\left.\left.+2n\left(x_1^{(0)}\gamma_1+x_2^{(0)}\gamma_2\right)+n^2\left(\gamma_1^2+\gamma_2^2\right)\right)\right)+\cdots=0.$$

Retaining the terms of the order $\hbar^{-2}$, we obtain their zero contribution

$$\left(-S_1^2-\dot{S}_0^2+\mathrm{i}\alpha x_2^{(0)}(\mathrm{i}\gamma_2\dot{S}_0+\mathrm{i}\gamma_1S_1)-\mathrm{i}\alpha x_1^{(0)}(\mathrm{i}\gamma_2S_1-\mathrm{i}\gamma_1\dot{S}_0)\right.$$
$$\left.+a^2-\frac{\alpha^2}{4}\left(\left(x_1^{(0)}\right)^2+\left(x_2^{(0)}\right)^2\right)\right)U=0,$$

as

$$\dot{S}_0(s)=a(s)-\frac{\alpha}{2}\left(x_1^{(0)}\gamma_1+x_2^{(0)}\gamma_2\right),$$
$$S_1(s)=\frac{\alpha}{2}\left(x_1^{(0)}\gamma_2-x_2^{(0)}\gamma_1\right).$$

Equating to zero all the terms of the order $\hbar^{-3/2}$ and taking into account that

$$\dot{S}_1=\frac{\alpha}{2}\left(1+\frac{x_1^{(0)}\gamma_1+x_2^{(0)}\gamma_2}{\rho}\right),$$

we obtain

$$a_1-2\alpha a-\frac{2}{\rho}a^2=0. \qquad \text{(A.1)}$$

The reason why this equation takes place must be clarified as follows. Classical trajectory of the particle with Hamiltonian

$$H = \frac{1}{2m}\left\{\left(p_1 + \frac{\alpha x_2}{2}\right)^2 + \left(p_2 - \frac{\alpha x_1}{2}\right)^2\right\} + u(\mathbf{x})$$

is described as an extremal solution of the action functional

$$S = \int \sqrt{2m(E - u(\mathbf{x}))}\,\mathrm{d}\sigma + \frac{\alpha}{2}\int(-x_2\dot{x}_1 + \dot{x}_2 x_1)\,\mathrm{d}\sigma.$$

In terms of the local coordinates s, n, related with the fixed trajectory $x_1^{(0)}(s), x_2^{(0)}(s)$, the action may be written as follows:

$$S = \int L(s, n, \dot{n})\,\mathrm{d}s,$$

where

$$\mathrm{d}\sigma = \sqrt{\left(1 - \frac{n(s)}{\rho(s)}\right)^2 + \dot{n}^2}\,\mathrm{d}s.$$

The integrand $L(s, n, \dot{n})$ is the corresponding Lagrangian, where the symbol $\dot{n}$ means a derivative of the extremal solution $n(s)$ with respect to s. Taking into account only asymptotically close trajectories $|n(s), \dot{n}(s)| \ll 1$ and using

$$u(\mathbf{x}(s)) = u_0(s) + u_1(s)n(s) + u_2(s)n(s)^2 + \cdots,$$

we may approximate the Lagrangian up to quadratic terms $L_2(s, n, \dot{n})$. The expression of $L_2(s, n, \dot{n})$ contains linear terms with respect to n and $\dot{n}$ with coefficients depending on s. The corresponding Euler–Lagrange equation

$$\frac{d}{\mathrm{d}s}\left(\frac{\partial L_2}{\partial \dot{n}}\right) = \frac{\partial L_2}{\partial n} \tag{A.2}$$

is homogeneous and given by

$$\ddot{n} + \dot{n}\frac{\dot{a}}{a} + nd(s) = 0, \tag{A.3}$$

if the following relation holds true:

$$\alpha + \frac{a}{\rho} + \frac{a u_1}{2(E - u_0)} = 0,$$

which is identical to (A.1). Equation (A.3) is being homogeneous means that $n(s) = 0$ must be a solution of (A.2) as well. Equation (A.3) is equivalent to the Hamiltonian system (11).

Finally, equating to zero all the terms of the order $\hbar^{-1}$ taking into account that

$$\ddot{S}_0 = \dot{a}(s) + \frac{1}{\rho}S_1$$

and introducing $\nu = n/\sqrt{\hbar} = O(1)$, we obtain

$$U_{\nu\nu} + 2\mathrm{i}aU_s + \mathrm{i}\dot{a}(s)U + \nu^2 a^2 U\left(\frac{a_2}{a^2} - \frac{a_1}{2\rho a^2} - \frac{1}{\rho^2} - \frac{a_1^2}{4a^4}\right) = 0.$$

The substitute

$$U = \frac{\psi_0(s, \nu)}{\sqrt{a(s)}}$$

leads to equation (8).

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